

Drag force on a sphere moving toward an anisotropic superhydrophobic planeEvgeny S. Asmolv,^{1,2,3} Aleksey V. Belyaev,^{1,4} and Olga I. Vinogradova^{1,4,5}¹*A.N. Frumkin Institute of Physical Chemistry and Electrochemistry, Russian Academy of Sciences, 31 Leninsky Prospekt, 119991 Moscow, Russia*²*Institute of Mechanics, M. V. Lomonosov Moscow State University, 119992 Moscow, Russia*³*Central Aero-Hydrodynamics Institute, 1 Zhukovsky str., Zhukovsky, Moscow region, 140180, Russia*⁴*Department of Physics, M. V. Lomonosov Moscow State University, 119991 Moscow, Russia*⁵*DWI, RWTH Aachen, Pauwelsstr. 8, D-52056 Aachen, Germany*

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We analyze theoretically a high-speed drainage of liquid films squeezed between a hydrophilic sphere and a textured superhydrophobic plane that contains trapped gas bubbles. A superhydrophobic wall is characterized by parameters L (texture characteristic length), b_1 and b_2 (local slip lengths at solid and gas areas), and ϕ_1 and ϕ_2 (fractions of solid and gas areas). Hydrodynamic properties of the plane are fully expressed in terms of the effective slip-length tensor with eigenvalues that depend on texture parameters and H (local separation). The effect of effective slip is predicted to decrease the force as compared with what is expected for two hydrophilic surfaces and described by the Taylor equation. The presence of additional length scales, L , b_1 , and b_2 , implies that a film drainage can be much richer than in the case of a sphere moving toward a hydrophilic plane. For a large (compared to L) gap the reduction of the force is small, and for all textures the force is similar to expected when a sphere is moving toward a smooth hydrophilic plane that is shifted down from the superhydrophobic wall. The value of this shift is equal to the average of the eigenvalues of the slip-length tensor. By analyzing striped superhydrophobic surfaces, we then compute the correction to the Taylor equation for an arbitrary gap. We show that at a thinner gap the force reduction becomes more pronounced, and that it depends strongly on the fraction of the gas area and local slip lengths. For small separations we derive an exact equation, which relates a correction for effective slip to texture parameters. Our analysis provides a framework for interpreting recent force measurements in the presence of a superhydrophobic surface.

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I. INTRODUCTION

Superhydrophobic Cassie (SH) surfaces have opened a whole new field of investigation, with both fundamental and practical perspectives [1]. They are able to trap air at the liquid-solid interface, leading to remarkable (“super”) properties, such as a very large water contact angle and low hysteresis. This strong hydrophobicity has macroscopic implications in the context of self-cleaning [2] and impact processes [3,4]. SH surfaces could also revolutionize microfluidic lab-on-a-chip systems [5,6] since the large effective slip of SH surfaces [7–9] compared to smooth hydrophobic channels [10–12] can greatly lower the viscous drag. SH surfaces can also amplify electrokinetic pumping [13–15] and mixing [9,16] in microfluidic devices.

In addition, this superlubrication potential should dramatically modify a squeeze film drainage between surfaces. In the previous study [17] we have analyzed a drag force F on a hydrophilic disk approaching a SH wall (the Reynolds problem). Here we explore what happens when a hydrophilic sphere of radius R is driven toward a SH plane with a velocity U (see Fig. 1); i.e., we address the so-called Taylor problem. Beside its significance as a geometry of surface forces apparatus (SFA) and atomic force microscope (AFM) force experiments, it represents a typical situation of phenomena of “viscous adhesion,” coagulation, and more. In a case of hydrophilic surfaces a hydrodynamic force reads [18]

$$F_T = \frac{6\pi\mu UR^2}{h} \quad (1)$$

when the gap $h \ll R$. Here μ denotes a fluid dynamic viscosity. In the case of a hydrophobic plane, characterized by a constant slip length b (the distance between the solid at which the flow profile extrapolates to zero) the Taylor expression has to be corrected for slip [19,20]:

$$f^* = \frac{F}{F_T} = \frac{1}{4} \left\{ 1 + \frac{3h}{2b} \left[\left(1 + \frac{h}{4b} \right) \times \ln \left(1 + \frac{4b}{h} \right) - 1 \right] \right\}. \quad (2)$$

The factor f^* associated with hydrodynamic slip can significantly decrease the hydrodynamic resistance force provided h is of the order of $4b$ or smaller. Equation (2) is often used to infer the value of a slip length from the force experiment. Indeed, the dynamic SFA/AFM force measurements [21] are extremely accurate at the nanoscale as compared to direct flow profiling or velocimetry [22]. Therefore, it is possible to measure even nanometric slip lengths [11,12,23,24]. The advantage of this (hydrophobic versus hydrophilic) geometry of configuration is that it allows us to avoid the gas bridging and long-range attractive capillary forces [25], that appear when we deal with interactions of two hydrophobic solids [26–28].

For these reasons it is attractive to consider the hydrodynamic interaction of a hydrophilic sphere with a SH surface. However, despite its importance for force experiments and numerous applications, the quantitative understanding of the problem is still challenging. The heterogeneous nature of the SH texture makes a precise discussion of the liquid flow past composite regions difficult, especially when surface patterns are anisotropic. It would seem therefore appropriate to bring a

more modern theoretical technique to bear on this problem. In this work we develop a theory that describes a force, acting on a sphere approaching the SH wall. Our theory is based on the effective slip approach introduced originally to describe flows past a single interface or in a flat channel [5,9,29,30]. In this approach, the effective slip is evaluated by averaging of a flow over the length scale of the experimental configuration being applied at the hypothetical smooth surface. Such a boundary condition mimics the actual one along the true heterogeneous SH surface, where gas pockets are stabilized with a rough wall texture. A corollary of this is that the effective hydrodynamic slip \mathbf{b}_{eff} generally depends on the flow direction being a tensor [29] and that it also depends on the separation between surfaces [9,17]. The concept of an effective tensorial slip has provided a great deal of insight into various factors that determine flow in a flat channel, and has allowed one to obtain very simple solutions of complex problems [9,29]. We shall see that it can be successfully applied to solve the Taylor problem in the presence of the SH surface. Our theory has the merit of yielding useful (approximate) analytical results as well as being very well suited to numerical work.

Our paper is arranged as follows: In Sec. II some general considerations concerning a description of a drainage of a liquid film confined between a hydrophilic sphere and arbitrary SH textured plane are presented. Here we also describe universal asymptotic solutions, valid for any texture. Section III contains analytical and numerical results for a striped SH surface. We conclude in Sec. IV with a discussion of our results and their possible relevance for force experiments. The Appendix contains a derivation of equations in a thin gap limit.

II. GENERAL THEORY

In this section we describe the theory of a hydrodynamic interaction of a sphere with an idealized SH surface in the Cassie state (sketched in Fig. 1), where a liquid slab lies on top of the surface roughness. The liquid-gas interface is assumed to be flat with no meniscus curvature, so that the modeled superhydrophobic surface appears as perfectly smooth with a pattern of boundary conditions. The latter are taken as low partial slip (b_1) over solid-liquid areas and as large partial slip (b_2) over gas-liquid regions. We denote as δ a typical length scale of gas-liquid areas. The fraction of solid-liquid areas will be denoted $\phi_1 = (L - \delta)/L$, and of gas-liquid areas $\phi_2 = 1 - \phi_1 = \delta/L$. In this idealization, some assumptions may have a possible influence on the friction properties and, therefore, a hydrodynamic force. First, by assuming a flat interface, we have neglected an additional mechanism for a dissipation connected with the meniscus curvature [30–32]. Second, we ignore a possible transition toward an impaled (Wenzel) state that can be provoked by additional pressure in the liquid phase [33,34].

The flow of liquid in the gap satisfies Stokes equations

$$\mu \frac{\partial^2 \mathbf{v}_\tau}{\partial z^2} \simeq \nabla_\tau p, \quad (3)$$

$$\frac{\partial v_z}{\partial z} + \nabla_\tau \cdot \mathbf{v}_\tau = 0, \quad (4)$$

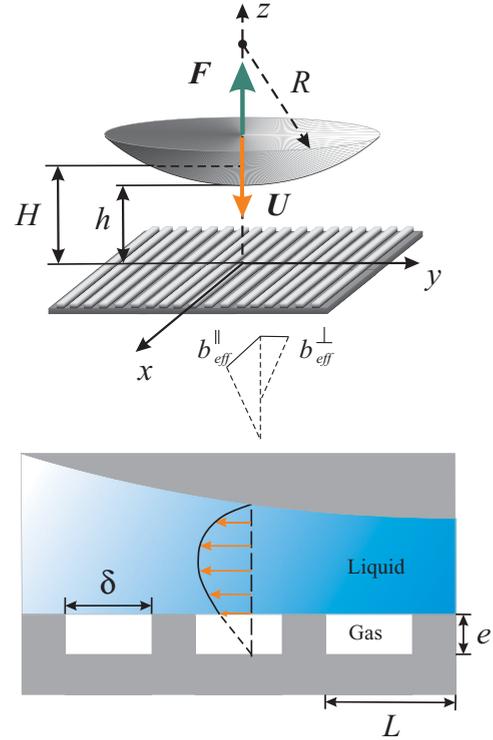


FIG. 1. (Color online) Sketch of a hydrophilic sphere approaching a striped superhydrophobic surface (top), and its model representation as a flat interface with patterns of boundary conditions (bottom).

where $\mathbf{v}_\tau = v_x \mathbf{e}_x + v_y \mathbf{e}_y$ is the lateral velocity, v_z is the normal velocity (directed toward the sphere), $p(x, y)$ is the local pressure, and ∇_τ is the differential operator in a plane (x, y) , given by

$$\nabla_\tau = \frac{\partial}{\partial x} \mathbf{e}_x + \frac{\partial}{\partial y} \mathbf{e}_y. \quad (5)$$

The SH plane (generally anisotropic) exhibits uniform tensorial slip [29], so that the boundary conditions are

$$z = 0: (v_\tau)_i = (b_{\text{eff}})_{ij} \frac{\partial (v_\tau)_j}{\partial z}, \quad v_z = 0; \quad (6)$$

$$z = H(x, y): \mathbf{v} = -U \mathbf{e}_z, \quad (7)$$

with summation over repeated indices. Near the axis the surface of a sphere can be approximated by a paraboloid of revolution

$$H = h + \frac{r^2}{2R} + O(r^4) \quad (r \ll R), \quad (8)$$

where $r^2 = x^2 + y^2$. The solution of Eq. (3) by imposing boundary conditions (6) and (7) yields

$$(v_\tau)_i = \frac{\nabla_j p}{2\mu} (z^2 \delta_{ij} - A_{ij} z - B_{ij}), \quad (9)$$

$$A_{ik} H + (b_{\text{eff}})_{ij} A_{jk} = H^2 \delta_{ik}, \quad B_{ik} = (b_{\text{eff}})_{ij} A_{jk},$$

where i and $j = \{x, y\}$.

For convenience, we now diagonalize the effective slip length tensor \mathbf{b}_{eff} , by aligning the x axis with the “fast” axis of greatest forward slip, $b_{\text{eff}}^{\parallel}$, which is always perpendicular to

the “slow” axis of least forward slip, b_{eff}^{\perp} [29]. By integrating Eq. (4) with Eqs. (6) and (7), and by using Eq. (9), we then derive a partial differential equation for pressure:

$$-\mu U = \frac{\partial}{\partial x} \left[H k_{\parallel}(H) \frac{\partial p}{\partial x} \right] + \frac{\partial}{\partial y} \left[H k_{\perp}(H) \frac{\partial p}{\partial y} \right], \quad (10)$$

where $k_{\parallel, \perp}$ are the permeabilities of a *flat* channel of thickness H with one SH wall

$$k_{\parallel, \perp} = \frac{H^2}{12} \left[\frac{H + 4b_{\text{eff}}^{\parallel, \perp}(H)}{H + b_{\text{eff}}^{\parallel, \perp}(H)} \right] = k \times k_{\parallel, \perp}^*.$$

Here $k = H^2/12$ represents an *isotropic* permeability of a flat hydrophilic channel with the same thickness, and $k_{\parallel, \perp}^*$ are corrections to this permeability due to SH slip.

The solution of Eq. (10) represents a very difficult problem since in a general case, a broken symmetry of the problem caused by a texture anisotropy does not allow one standard simplifications, such as representing $p(x, y)$ as $p(H)$ [20,36], etc. Since $k_{\parallel, \perp}$ are radially symmetric, it is more convenient to solve Eq. (10) by using polar coordinates:

$$\begin{aligned} -\mu U = & \frac{\partial}{\partial r} \left(H \langle k \rangle \frac{\partial p}{\partial r} \right) + \frac{H \langle k \rangle}{r} \frac{\partial p}{\partial r} + \frac{H \langle k \rangle}{r^2} \frac{\partial^2 p}{\partial \varphi^2} \\ & + \cos 2\varphi \left[\frac{\partial}{\partial r} \left(H \Delta k \frac{\partial p}{\partial r} \right) - \frac{H \Delta k}{r} \frac{\partial p}{\partial r} - \frac{H \Delta k}{r^2} \frac{\partial^2 p}{\partial \varphi^2} \right] \\ & - \frac{\sin 2\varphi}{r} \left[2H \Delta k \frac{\partial^2 p}{\partial r \partial \varphi} + \left(\frac{d(H \Delta k)}{dr} - \frac{2H \Delta k}{r} \right) \frac{\partial p}{\partial \varphi} \right], \end{aligned} \quad (11)$$

where

$$\langle k \rangle(r) = \frac{k_{\parallel} + k_{\perp}}{2}, \quad \Delta k(r) = \frac{k_{\parallel} - k_{\perp}}{2},$$

with boundary conditions

$$p(R, \varphi) = 0, \quad \frac{\partial p}{\partial r}(0, \varphi) = 0. \quad (12)$$

The condition for a pressure takes the form $p(r \rightarrow \infty) = 0$, provided $R \gg \max\{b, L, h\}$.

In general case, the solution of Eq. (11) is not radially symmetric, $\Delta k \neq 0$, due to the terms proportional to $\cos 2\varphi$ and $\sin 2\varphi$. However, it is symmetric with respect to x and y axes; hence, it can be presented in terms of a cosine series:

$$p = p_0(r) + p_1(r) \cos 2\varphi + p_2(r) \cos 4\varphi + \dots \quad (13)$$

This expansion can then be used to solve Eq. (11) numerically. Also an asymptotic solution can be constructed, provided $\varepsilon = \Delta k(0)/\langle k \rangle(0) \ll 1$ [37].

The hydrodynamic resistance force is given by

$$F(h) = F_T f^* = \int_0^\infty \int_0^{2\pi} p r d\varphi dr = 2\pi R \int_h^\infty p_0 dH. \quad (14)$$

We stress that terms with $\cos(2n\varphi)$ vanish after integration over φ in the interval $[0, 2\pi]$, so that only *isotropic* part of the pressure, p_0 , contributes to the drag force.

In two limiting cases, namely, small ($H \ll L$) and large ($H \gg L$) distances between surfaces, the pressure distribution could be treated with high accuracy as radially symmetric

[37]. Once such assumption is made, the governing equation Eq. (11) for pressure could be rewritten in the form

$$-\mu U = \frac{d}{dr} \left(H \langle k \rangle \frac{d\tilde{p}}{dr} \right) + \frac{H \langle k \rangle}{r} \frac{d\tilde{p}}{dr}. \quad (15)$$

Here we use tilde to distinguish approximate \tilde{p} from p_0 , which is an isotropic part of a general solution, Eq. (13). Equation (15) can be then integrated to give

$$\frac{d\tilde{p}}{dr} = -\frac{\mu U r}{2H \langle k \rangle} = -\frac{12\mu U r}{H^3 [k_{\parallel}^*(H) + k_{\perp}^*(H)]}, \quad (16)$$

so that we can represent $p(x, y)$ as a function of $H(x, y)$:

$$\frac{\tilde{p}(H)}{12\mu U R} = \int_H^\infty \frac{dH'}{H'^3 (k_{\parallel}^* + k_{\perp}^*)}. \quad (17)$$

The hydrodynamic resistance force then can be calculated as

$$F = F_T f^* = 2\pi R \int_h^\infty \tilde{p} dH. \quad (18)$$

In a general case, the pressure and the force should be found numerically, and their calculations require detailed knowledge about eigenvalues of an effective slip tensor for some particular texture. However, in some situations asymptotic solutions could be obtained, and they are the same for all textures. Below we briefly discuss some of them.

For a *large* gap, $H \gg L$, the corrections to permeability are [29]

$$k_{\parallel}^* \simeq 1 + \frac{3b_{\text{eff}}^{\parallel}}{H}, \quad k_{\perp}^* \simeq 1 + \frac{3b_{\text{eff}}^{\perp}}{H}. \quad (19)$$

Even for strongly slipping ($b_2 \gg L$) surfaces, the effective slip lengths remain independent of the gap thickness and are small compared to it [17]. Consequently,

$$\begin{aligned} \langle k \rangle &= 1 + \frac{3(b_{\text{eff}}^{\parallel} + b_{\text{eff}}^{\perp})}{H} + O(\varepsilon^2), \\ \Delta k &= \frac{3(b_{\text{eff}}^{\parallel} - b_{\text{eff}}^{\perp})}{H} [1 + O(\varepsilon)], \end{aligned}$$

so that $\Delta k/\langle k \rangle \simeq b_{\text{eff}}^{\parallel}/H \ll 1$. The solution of Eq. (11) can then be constructed in terms of a series of ε . In this case an asymmetric part of the pressure field is small, and the radially symmetric part may be approximated by \tilde{p} with a very high accuracy, $p_0 = \tilde{p}[1 + O(\varepsilon^2)]$ [37]. The correction to SH slip is then

$$f^* \simeq 1 - \frac{b_{\text{eff}}^{\parallel} + b_{\text{eff}}^{\perp}}{2h}, \quad (20)$$

which coincides with the result, obtained earlier for corrugated (Wenzel) surfaces via the Lorentz reciprocal theorem [38]. Physically, Eq. (20) means that the anisotropic SH plane at a distance h from the sphere apex is equivalent to a no-slip plane at a distance $h + (b_{\text{eff}}^{\parallel} + b_{\text{eff}}^{\perp})/2$. The shift of this equivalent no-slip plane from the real SH surface is thus equal to the average of the eigenvalues of the effective slip-length tensor.

For weakly slipping surfaces, $b_2 \ll L$, Eq. (20) can be further simplified since in this situation the effective slip-length

tensor is isotropic, and the effective slip coincides with the surface average [30,39]. Therefore,

$$f^* \simeq 1 - \frac{b_1\phi_1 + b_2\phi_2}{h}. \quad (21)$$

For a *thin* gap, $H \ll L$, and weakly slipping surfaces, $b_2 \ll H$, the flow is also isotropic [8], and corrections to permeabilities are given by Eq. (19). In what follows corrections for SH slip are given by Eq. (21). In other situations f^* is sensitive to the pattern geometry and should be calculated individually for a particular texture of interest.

III. STRIPED SUPERHYDROPHOBIC SURFACES

To illustrate the predictions of the general theory, we now focus on a flat, periodic, striped SH surface. This canonical texture is convenient to explore the basic physics of the system since the local (scalar) slip lengths, b_1 and b_2 , vary only in one direction, thus allowing us to highlight effects of anisotropy. The problem of flow past striped SH surfaces has previously been studied in a context of a reduction of pressure-driven forward flow in thick [39–41] and thin [8] channels, and it is directly relevant for a passive mixing [9,16] and a generation of a tensorial electro-osmotic flow [9,14,15].

A. Arbitrary gap, nonslipping solid regions

The striped SH surface with $b_1 = 0$ currently seems to be the only one where the eigenvalues of the slip length tensor have been calculated for an arbitrary thickness of the channel [9]. For the general case the eigenvalues can be calculated semianalytically, but accurate analytical results have been found in the thin and thick channel limits.

In the case of thick channels, $H \gg L$, the eigenvalues of the slip-length tensor read [39]

$$b_{\text{eff}}^{\parallel} \simeq \frac{L}{\pi} \frac{\ln \left[\sec \left(\frac{\pi\phi_2}{2} \right) \right]}{1 + \frac{L}{\pi b_2} \ln \left[\sec \left(\frac{\pi\phi_2}{2} \right) + \tan \left(\frac{\pi\phi_2}{2} \right) \right]}, \quad (22)$$

$$b_{\text{eff}}^{\perp} \simeq \frac{L}{2\pi} \frac{\ln \left[\sec \left(\frac{\pi\phi_2}{2} \right) \right]}{1 + \frac{L}{2\pi b_2} \ln \left[\sec \left(\frac{\pi\phi_2}{2} \right) + \tan \left(\frac{\pi\phi_2}{2} \right) \right]}. \quad (23)$$

Flow in a large channel does not depend on H and is controlled by the ratio of the local slip length b_2 to texture period L . When $b_2/L \ll 1$, Eqs. (22) and (23) predict the area-averaged isotropic slip length, $b_{\text{eff}}^{\perp} \simeq \phi_2 b_2$. When $b_2/L \gg 1$, expressions (22) and (23) take the form

$$b_{\text{eff}}^{\perp} \simeq \frac{L}{2\pi} \ln \left[\sec \left(\frac{\pi\phi_2}{2} \right) \right], \quad b_{\text{eff}}^{\parallel} \simeq 2b_{\text{eff}}^{\perp}, \quad (24)$$

which coincides with results obtained for the perfect slip ($b_2 = \infty$) stripes [40].

In the case of thin channels, $H \ll L$, striped surfaces have been shown to provide rigorous upper and lower Wiener bounds on the effective slip over all possible two-phase patterns [8]:

$$b_{\text{eff}}^{\parallel} = \frac{b_2 H \phi_2}{H + b_2 \phi_1}, \quad b_{\text{eff}}^{\perp} = \frac{b_2 H \phi_2}{H + 4b_2 \phi_1}. \quad (25)$$

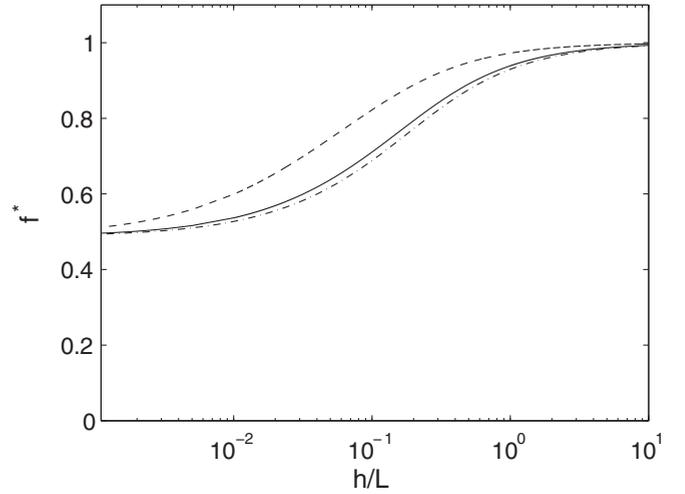


FIG. 2. A correction to the resistance force for SH slip, f^* , as a function of h/L , calculated with $b_1 = 0$ and $b_2/L = 10$ (dash-dots), 1 (solid curve), and 0.1 (dashed curve). The fraction of gas regions is $\phi_2 = 0.5$.

At $b_2/H \gg 1$ these give truly tensorial anisotropic effective slip,

$$b_{\text{eff}}^{\parallel} = H \frac{\phi_2}{\phi_1}, \quad b_{\text{eff}}^{\perp} = \frac{b_{\text{eff}}^{\parallel}}{4}, \quad (26)$$

but at $b_2/H \ll 1$ it leads again to a surface average slip.

Calculations of b_{eff}/H , made for several b_2/L , were used to compute pressure, and then the correction for effective slip f^* as a function of the gap (Fig. 2). For this numerical example we have chosen $\phi_2 = 0.5$, which gives a maximum transverse flow in a thin channel situation [16]. All calculations were performed on the assumption of the radially symmetric pressure field, $p = \bar{p}(H)$, which should give reasonably accurate results even at the intermediate distances, i.e., when the effective slip changes rapidly with H [9,37]. Figure 2 shows that at large distances all curves converge to $f^* = 1$; i.e., the drag force is the same that it would be in the case of a hydrophilic plane, $F = F_T$. This conclusion directly follows from Eq. (20) and is valid for any, however large, b_2 . The correction for SH slip significantly decreases when h becomes on the order of L and smaller. It can be seen that the increase in b_2 leads to a smaller drag force at large and intermediate distances. However, at $h \ll L$ all curves tend to a constant, which does not depend on the value of the local slip length at the gas sectors.

Figure 3 includes theoretical curves calculated for $b_2/L = 10$ with different fractions of the surface gas phase. Results show that the correction for effective slip has a tendency to decrease with ϕ_2 , and that for each ϕ_2 there is some minimum value of f^* that all curves approach asymptotically. We are now in a position to quantify this important result. When $b_1 = 0$ and $h \ll \min\{b_2, L\}$ corrections to permeabilities take a form [8]

$$k_{\parallel}^* = 1 + 3\phi_2, \quad k_{\perp}^* = \frac{4}{4 - 3\phi_2}; \quad (27)$$

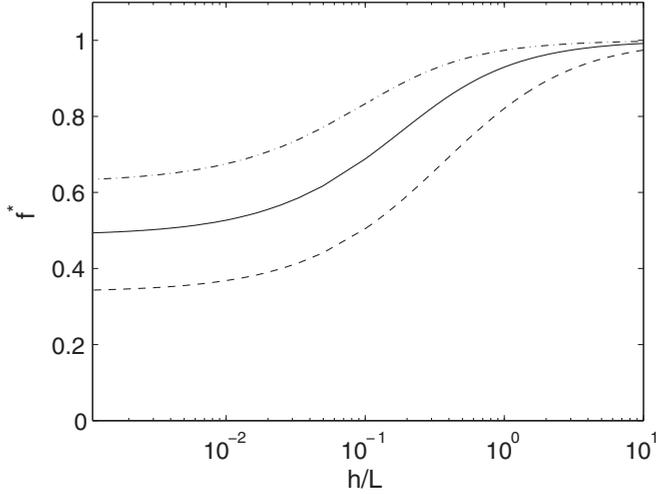


FIG. 3. A correction for SH slip to the resistance force exerted on a sphere moving toward the anisotropic plane of stripes with no slip on solid-liquid regions ($b_1 = 0$). $\phi_2 = 0.3$ (dash-dotted), 0.5 (solid), 0.8 (dash), $b_2/L = 10$.

i.e., $k_{\parallel}^*/k_{\perp}^*$ is independent of H , and \tilde{p} coincides with the exact solution of Eq. (11). Therefore,

$$\frac{\tilde{p}(H)}{3\mu UR} = \frac{2(4 - 3\phi_2)}{(8 + 9\phi_2 - 9\phi_2^2)H^2} \quad (28)$$

and

$$f^* = \frac{2(4 - 3\phi_2)}{8 + 9\phi_2 - 9\phi_2^2}. \quad (29)$$

This result coincides with the expression obtained earlier for an interaction of a disk with a similar SH surface [17] and shows that in the limit of a thin gap f^* depends only on the surface fraction of the gas phase.

Since ϕ_2 is one of the key parameters determining reduction of drag, in Fig. 4 we plot the correction for SH slip as a function of ϕ_2 at several h/L . We see that when ϕ_2 is very small, the correction for SH slip tends to its absolute maximum, $f^* = 1$. In the most interesting limit, $\phi_2 \rightarrow 1$, we can reach the minimum possible value of correction for SH slip, $f^* \rightarrow 1/4$, provided b_2/h is large.

B. Thin gap, slipping solid regions

In the previous subsection we ignored the (small) slip length at the solid areas. The finite b_1 cannot modify our conclusions made for a thick or intermediate gap. It could, however, influence a thin gap situation. Fortunately, in the limit of a thin channel, the exact values of longitudinal and transverse permeabilities are known for any piecewise constant local slip length, b_1 and b_2 . These values represent bounds that constrain the attainable effective permeabilities and allow us to calculate effective slip lengths [8], and then f^* for arbitrary local slip lengths.

The expression for a longitudinal permeability can be presented as [8]

$$k_{\parallel} = \phi_1 k_1 + \phi_2 k_2 = k \times k_{\parallel}^*, \quad (30)$$

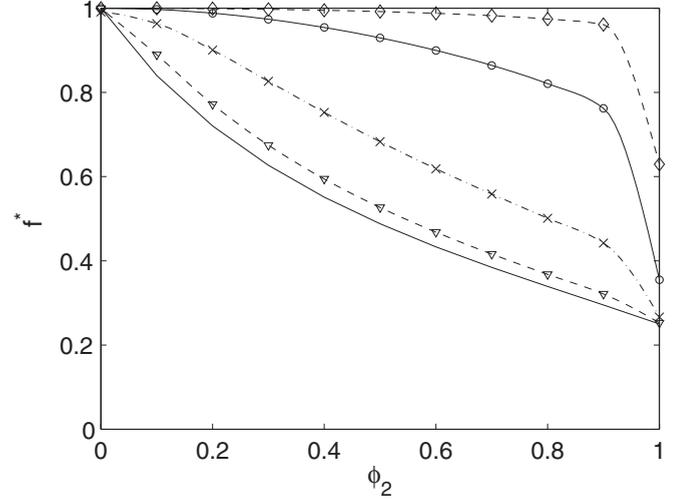


FIG. 4. A correction for SH slip as a function of a fraction of the gas phase at $b_1 = 0$ and $b_2/L = 10$. From top to bottom, curves with symbols correspond to $h/L = 0.01, 0.1, 1$, and 10. Solid line without symbols shows calculation results obtained in the limit of a thin gap, $h \ll \min\{b_2, L\}$.

with

$$k_{\parallel}^* = \phi_1 \left(\frac{H + 4b_1}{H + b_1} \right) + \phi_2 \left(\frac{H + 4b_2}{H + b_2} \right). \quad (31)$$

Transverse stripes, in turn, satisfy the equation

$$k_{\perp} = \left(\frac{\phi_1}{k_1} + \frac{\phi_2}{k_2} \right)^{-1} = k \times k_{\perp}^*, \quad (32)$$

where

$$k_{\perp}^* = \left(\phi_1 \frac{H + b_1}{H + 4b_1} + \phi_2 \frac{H + b_2}{H + 4b_2} \right)^{-1}. \quad (33)$$

These expressions allow one to calculate effective slip lengths in eigendirections

$$b_{\text{eff}}^{\parallel} = \frac{H\phi_1 b_1 + H\phi_2 b_2 + b_1 b_2}{H + \phi_2 b_1 + \phi_1 b_2}, \quad (34)$$

$$b_{\text{eff}}^{\perp} = \frac{H\phi_1 b_1 + H\phi_2 b_2 + 4b_1 b_2}{H + 4\phi_2 b_1 + 4\phi_1 b_2}. \quad (35)$$

In general case, the pressure and the force should be calculated numerically. Some special cases, however, allow one exact analytical solutions. In particular, when $H \ll \min\{b_2, L\}$ the corrections to permeabilities due to SH effective slip become

$$k_{\parallel}^*(H) \simeq \frac{H(1 + 3\phi_2) + 4b_1}{H + b_1}, \quad (36)$$

$$k_{\perp}^*(H) \simeq \frac{4(H + 4b_1)}{4(H + b_1) - 3\phi_2 H}. \quad (37)$$

The expressions for pressure and force in this situation are rigorously derived in the Appendix, and an exact analytical expression for f^* is given by Eq. (A2). Figure 5 includes theoretical curves, f^* versus ϕ_2 , calculated for several b_1/h . At $\phi_2 \rightarrow 0$, and $\phi_2 \rightarrow 1$, our results are consistent with earlier predictions for homogeneous hydrophobic solids, with b_1 and

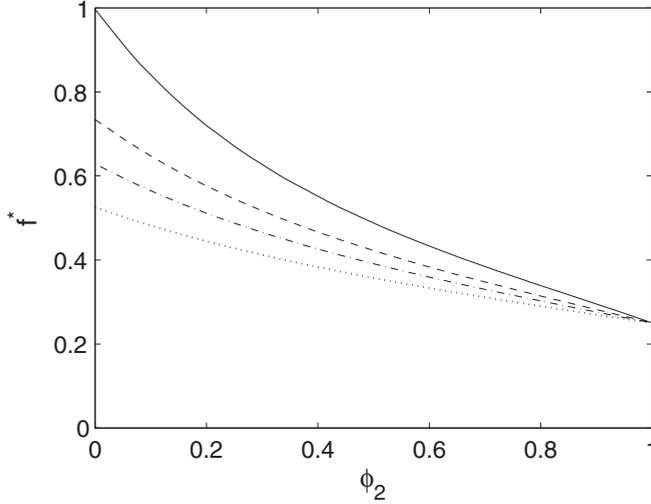


FIG. 5. The correction for SH slip f^* vs surface fraction of the gas phase ϕ_2 in the limit of a thin gap and for $b_2/h \gg 1$. From top to bottom, $b_1/h = 0, 0.5, 1$, and 2 .

b_2 , described by Eq. (2) [19]. In general, a finite hydrophobic slippage at the solid sectors can significantly reduce f^* at short distances.

IV. FINAL REMARKS

Certain aspects of our work warrant further comments. We have presented data describing the squeeze-film drainage of liquid in thin films between a hydrophilic sphere and a SH plane. The predicted drag force is smaller than expected for two hydrophilic solids and shows much more complex behavior as compared with what is expected in the case of a uniform hydrophobic wall. For convenience, we now summarize our main results for a striped SH surface in Table I. At first sight it is somewhat surprising that a system with one SH surface should exhibit such rich hydrodynamic properties. However, it becomes almost self-evident (from the presence of extra fundamental length scales, such as L , and two local slip lengths) that it should exhibit richer behavior than a geometry with hydrophilic or even homogeneous hydrophobic surfaces. Do our results have any consequences for force experiments with real SH surfaces?

TABLE I. Some useful asymptotic expressions for the correction factor f^* in the case of a striped superhydrophobic surface.

Case	f^*
$h \ll \min\{b_2, L\}$	Eq. (A2)
$b_1 = 0, h \ll \min\{b_2, L\}$	$\frac{2(4 - 3\phi_2)}{8 + 9\phi_2 - 9\phi_2^2}$
$L \gg h \gg b_2$	$1 - \frac{b_1\phi_1 + b_2\phi_2}{h}$
$h \gg L$	$1 - \frac{b_{\text{eff}}^{\parallel} + b_{\text{eff}}^{\perp}}{2h}$,
	with Eqs. (22) and (23)

Hydrodynamic force can be measured using various techniques, such as AFM, SFA, and others. The geometry of configuration is equivalent to a sphere, of a large radius of curvature, approaching a flat surface. Hydrodynamic forces at small (compared to R) separations have been already reported for textured isotropic periodic [42] and random [24] SH surfaces. Also we are not aware of any measurements of the hydrodynamic force for the striped SH surface; measurements with isotropic textures lend some support to the picture of the significant reduction of a hydrodynamic force presented here. It would appear that reduction of a hydrodynamic drag that is observed is consistent with an earlier theoretical model, Eq. (2), that gives b_{eff} on the order of 50–200 nm [24,42], which is only slightly larger than observed at smooth hydrophobic solids [10,11,23]. We suggest that further analysis of these measurements or similar measurements with other SH textures should employ one of the equations given in Table I rather than Eq. (2) since the latter makes assumptions of a constant, isotropic, homogeneous, and independent of distance slip that are not generally valid for a heterogeneous SH surface. While we maintain that Eq. (2) is the appropriate formula for describing f^* at very small ($\phi_2 \rightarrow 0$) and very large ($\phi_2 \rightarrow 1$) fractions of the gas phase, we note that other formulas will be required in all different situations with the regime of validity given in Table I.

From our calculations it is evident that simple far-field Eq. (20) gives an accurate estimate of f^* and should certainly be used to deduce the average of eigenvalues of \mathbf{b}_{eff} at large separations. We note that similar evaluations could be used for *rough* or *porous* surfaces since at large distances from the wall the boundary condition at the rough interface or fluid-porous interface may be approximated by a slip model [38,43–45]. It should be possible to make such an analysis of recent data obtained with grooved surfaces [46], which we suspect will significantly alter the conclusion of these authors. Note that the average eigenvalues of an effective slip length tensor at large (compared to L) separations should coincide with the measured by velocimetry [47] or other far-field methods [48]. We remark and stress, however, that the near-field effective slip often measured in AFM/SFA should be much smaller than a far-field slip. This is exactly what has been observed in experiment [24,42].

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APPENDIX: ANALYTIC SOLUTION FOR \tilde{p} AND f^* IN THE THIN-CHANNEL LIMIT

In this Appendix we derive analytical expressions for a dimensionless pressure distribution $\tilde{P} = \tilde{p}h^2(3\mu UR)^{-1}$ and for f^* in a situation when $h \ll \min\{b_2, L\}$, and the corrections to effective permeabilities are given by Eqs. (36) and (37).

We use the substitution into Eq. (17):

$$x = 4 - \frac{\eta' + 4\beta}{\eta' + \beta}, \quad \eta' = \frac{H'}{h}, \quad \beta = \frac{b_1}{h},$$

to obtain

$$\begin{aligned} \tilde{P}(\eta) &= 4 \int_{\eta}^{\infty} \frac{d\eta'}{\eta'^3(k_{\parallel}^* + k_{\perp}^*)} = 4 \int_{\frac{3\eta}{\eta+\beta}}^3 \frac{(3-x)(4-\phi_2 x) dx}{x^3[\phi_1 \phi_2 (4-x)^2 + 4(\phi_1^2 + \phi_2^2 + 1)(4-x) + 16\phi_1 \phi_2]} \\ &= \frac{1}{4} \left[-\frac{3}{x^2} + \frac{(4-3\phi_1)}{2x} + \beta^2 c_1 \ln \left| \frac{x^2}{\phi_1 \phi_2 x^2 - 8x + 32} \right| + \beta^2 c_2 \ln \left| \frac{\phi_1 \phi_2 x - 4 - d}{\phi_1 \phi_2 x - 4 + d} \right| \right] \Bigg|_{\frac{3\eta}{\eta+\beta}}^3 \\ &= \frac{3}{4} \left[\frac{1}{3\eta^2} + \frac{\phi_1}{2\beta\eta} + c_1 \ln \left(\frac{\eta^2 + a_1\eta + a_2}{\eta^2} \right) + c_2 \ln \left(\frac{\eta + a_3}{\eta + a_4} \right) \right], \end{aligned} \quad (\text{A1})$$

$$\begin{aligned} c_1 &= \frac{\phi_1(3\phi_1 - 5)}{32\beta^2}, \quad c_2 = \frac{\phi_1(3\phi_1 - 1)(2\phi_1 - 3)}{8\beta^2 a_6}, \quad a_1 = \frac{40\beta}{a_5}, \quad a_2 = \frac{32\beta^2}{a_5}, \quad a_3 = \frac{\beta(a_6 - 4)}{3\phi_1 \phi_2 + a_6 - 4}, \\ a_4 &= -\frac{\beta(a_6 + 4)}{3\phi_1 \phi_2 - a_6 - 4}, \quad a_5 = 8 + 9\phi_2 - 9\phi_2^2, \quad a_6 = 4\sqrt{1 - 2\phi_2 + 2\phi_2^2}. \end{aligned}$$

Then one can also integrate analytically Eq. (14) with \tilde{P} given by (A1) to get

$$\begin{aligned} f^* &= \int_1^{\infty} \tilde{P} d\eta = \frac{3}{4} \left\{ -\frac{1}{3\eta} + \frac{\phi_1}{2\beta} \ln \eta + c_1 \left[\left(\eta + \frac{a_1}{2} \right) \ln(\eta^2 + a_1\eta + a_2) + a_7 \ln \left(\frac{2\eta + a_1 + 2a_7}{2\eta + a_1 - 2a_7} \right) - a_1 - 2\eta \ln \eta \right] \right. \\ &\quad \left. + c_2 [(\eta + a_3) \ln(\eta + a_3) - (\eta + a_4) \ln(\eta + a_4)] \right\} \Bigg|_1^{\infty} = \frac{3}{4} \left\{ \frac{1}{3} - c_1 \left[\left(1 + \frac{a_1}{2} \right) \ln(1 + a_1 + a_2) - a_1 \right] \right. \\ &\quad \left. + [c_1 a_7 + c_2(1 + a_4)] \ln(1 + a_4) - [c_1 a_7 + c_2(1 + a_3)] \ln(1 + a_3) - c_2(a_4 - a_3) \right\}, \end{aligned} \quad (\text{A2})$$

$$a_7 = 3\beta a_6 / a_5.$$

Equation (A2) reduces to Eq. (2) as $\phi_2 \rightarrow 0$ and to Eq. (29) as $\beta \rightarrow 0$.

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